Results in Engineering 5 (2020) 100062

Contents lists available at ScienceDirect

Results in Engineering

journal homepage: www.editorialmanager.com/rineng/Default.aspx

Analysis of a laterally load rigid cylinder embedded in an elastoplastic material

Salma F. Hashem Ali, Ashraf S. Osman *

A B S T R A C T

An analytical approach has been developed to predict the response of a rigid disc embedded in elastoplastic media. The governing differential equations are obtained using the principle of minimizing the potential energy. The displacement components in cylindrical coordinates could be represented by Fourier series. The fitted method is used to determine the Fourier series harmonic coefficients. Validation is made against finite element analysis and previously published solutions.

Introduction

In the practical design of pile foundations, the calculations related to the determination of the dimensions of piles are divided into two distinct groups: the stability calculations and the serviceability problems. They are treated in two separate and unrelated ways. Elasticity theory is used for displacement calculations while plasticity theory is used to evaluate the stability of structures.

Conventionally stability calculations of piles subjected to horizontal loads are based primarily on calculations at failure. In these methods, the soil is taken to be everywhere in a state of failure under ultimate horizontal load. The failure load of a laterally loaded pile was first treated in detail by Broms [1], who related the force per unit length P on the pile to the strength of the soil k. For cohesive soils, a constant limiting pressure is assumed at the soil-pile interface. Close to the ground surface, the limiting values were modified to allow for the different mode of deformation. Brom's approach was largely empirical with no theoretical justification.

Plane-strain conditions are often assumed in the analysis of laterally loaded piles, and the pile is modelled as a rigid disc. Randolph and Houlsby [2] used the limit theorems of plasticity to calculate upper and lower bounds to collapse load of a laterally loaded disc. The upper bound solution is based on kinematically admissible velocity fields with discontinuities. An alternative velocity field with discontinuities is introduced by Martin and Randolph [3] who were able were able to bracket the exact collapse load with a difference less than 0.65% between the upper and lower bound solutions. Klar and Osman [4] developed a continuous flow field for upper bound calculations of laterally loaded rigid discs with different soil/disc interface.

In pile displacement calculations, Winkler's [5] concept of the modulus of subgrade reaction is often used in the design. These methods compare the pile to a beam on elastic supports, and soil reaction is assumed to be proportional to lateral deformations and the modulus of subgrade reaction taken as the coefficient of proportionality. In more recent work, the moduli of reaction are defined as the slope of the pressure-deflexion curve. The latter approach is more useful as the subgrade modulus is similar, both in dimensions and value, to the stiffness of the soil.

An attractive alternative to Winkler analysis for displacement of laterally loaded piles is the use of continuum elastic approach based on variational method. This approach was first introduced by Vlasov and Leontiev [6] for analysis of beam resting on elastic layer. Vlasov variational approach has been used to analysis shallow foundations (see for examples [7–9]). It was then extended to pile displacement analysis [10–13]. In Vlasov variational approach, the displacements are expressed as a multiplication of one-dimensional functions. For example, in the 3D analysis of a laterally loaded pile in cylindrical coordinates (r,θ,z), the radial displacement U and the circumferential displacement V can be expressed by:

$$\begin{aligned} U &= F(z)\psi_1(r,\theta) = F(z)u(r)\cos\theta \\ V &= F(z)\psi_2(r,\theta) = -F(z)v(r)\sin\theta \end{aligned} \tag{1}$$

where u(r) and v(r) are functions of r only and F(z) is the deflexion of the pile.

The axial displacement of the pile is ignored as it is significantly small compared with the radial and circumferential displacement. The key advantage of the variational approach is that the solution of two and three-dimensional problems is obtained by solving systems of one-dimensional equations (as it will be demonstrated later). Therefore, simplified solutions could be achieved. The Vlasov variational analysis is not restricted to foundations problems, but it could be applied to various boundary-value problems. For example, Osman and Birchall [14] used

* Corresponding author. Department of Engineering, Durham University, South Road, Durham, DH1 3LE, UK.
E-mail address: ashraf.osman@durham.ac.uk (A.S. Osman).

https://doi.org/10.1016/j.rineng.2019.100062
Received 25 June 2019; Received in revised form 9 November 2019; Accepted 14 November 2019

this approach to analyse tunnels deeply embedded in viscoelastic soil.

The main limitation of Vlasov variational approach and Winkler methods is that they are based on the assumption that the soil is an elastic material. It is well established that soil exhibits plasticity behaviour even at a small strain level. This note presents the first attempt to extend Vlasov variational methods to elastoplastic problems.

It should be noted that in Vlasov variational analysis of laterally loaded piles the terms $u(r)\cos\theta$ and $-v(r)\sin\theta$ in equation (1) represents radial and the circumferential deformation at any horizontal plane along the pile normalised by the equivalent pile deflexion, and they are uniform along the pile. Therefore, plane-strain simplification is justified because the displacements of the various sections of the pile vary in a continuous manner. In this note, plane-strain conditions are assumed, and the authors present an elastoplastic analysis of a rigid disc embedded in an elastoplastic material and subjected to a lateral load using the variation method.

Formulation of elastoplastic solution

The displacement components in cylindrical coordinates could be represented by Fourier series expressed as:

$$U=\sum_{k=0}^{L} u_k{}^{(1)}(r)\cos k\theta+\sum_{k=1}^{L} u_k{}^{(2)}(r)\sin k\theta \tag{2}$$

$$V=-\sum_{k=1}^{L} v_k{}^{(1)}(r)\sin k\theta+\sum_{k=0}^{L} v_k{}^{(2)}(r)\cos k\theta \tag{3}$$

where U and V are the radial and circumferential incremental displacements, and $u_k{}^{(1)}$, $u_k{}^{(2)}$, $v_k{}^{(1)}$ and $v_k{}^{(2)}$ are the 0^{th} and L^{th} order cosine and L^{th} order sine harmonic coefficients of variables U and V. The total strain can be found from the first derivative of displacements as

$$\varepsilon=\begin{Bmatrix}\varepsilon_{rr}\\ \varepsilon_{\theta\theta}\\ \varepsilon_{zz}\\ 2\varepsilon_{r\theta}\end{Bmatrix}=\begin{Bmatrix}\dfrac{dU}{dr}\\ \dfrac{1}{r}\left(U+\dfrac{dV}{d\theta}\right)\\ 0\\ \dfrac{1}{r}\dfrac{dU}{d\theta}+\dfrac{dV}{dr}-\dfrac{V}{r}\end{Bmatrix} \tag{4}$$

where ε_{ij} denotes a strain component and ε is a total strain.

An elastic strain can be calculated as:

$$\varepsilon^e=\varepsilon-\varepsilon^p \tag{5}$$

ε^p represents plastic strain which could also be expressed by Fourier series as:

$$\varepsilon_r{}^p=\sum_{k=0}^{L}\widehat{\varepsilon}_{rk}{}^{p(1)}(r)\cos k\theta+\sum_{k=1}^{L}\widehat{\varepsilon}_r{}^{p(2)}(r)\sin k\theta \tag{6}$$

$$\varepsilon_\theta{}^p=\sum_{k=0}^{L}\widehat{\varepsilon}_{\theta k}{}^{p(1)}(r)\cos k\theta+\sum_{k=1}^{L}\widehat{\varepsilon}_{\theta k}{}^{p(2)}(r)\sin k\theta \tag{7}$$

$$\varepsilon_z{}^p=\sum_{k=0}^{L}\widehat{\varepsilon}_{zk}{}^{p(1)}(r)\cos k\theta+\sum_{k=1}^{L}\widehat{\varepsilon}_{zk}{}^{p(2)}(r)\sin k\theta \tag{8}$$

$$\varepsilon_{r\theta}{}^p=-\sum_{k=1}^{L}\widehat{\varepsilon}_{r\theta k}{}^{p(1)}(r)\sin k\theta+\sum_{k=0}^{L}\widehat{\varepsilon}_{r\theta k}{}^{p(2)}(r)\cos k\theta \tag{9}$$

In plane strain, elastic stresses can be calculated by multiplying the elastic strain vector with the stiffness matrix:

$$\{\sigma\}=\begin{Bmatrix}\sigma_{rr}\\ \sigma_{\theta\theta}\\ \sigma_{zz}\\ \sigma_{r\theta}\end{Bmatrix}=[D]\{\varepsilon^e\}$$

$$=\begin{bmatrix}\lambda+2G & \lambda & \lambda & \\ \lambda & \lambda+2G & \lambda & \\ \lambda & \lambda & \lambda+2G & \\ & & & G\end{bmatrix}\begin{Bmatrix}\dfrac{dU}{dr}-\varepsilon_{rr}{}^p\\ \dfrac{1}{r}\left(U+\dfrac{dV}{d\theta}\right)-\varepsilon_{\theta\theta}{}^p\\ 0-\varepsilon_{zz}{}^p\\ \dfrac{1}{r}\dfrac{dU}{d\theta}+\dfrac{dV}{dr}-\dfrac{V}{r}-2\varepsilon_{r\theta}{}^p\end{Bmatrix} \tag{10}$$

where $[D]$ is the stiffness matrix, λ and G are Lame constants which could be taken to be constant for linear soil.

The total potential energy can be expressed as

$$\prod=F+D \tag{11}$$

where F denotes the free energy and D is dissipated energy.

Equation (10) can be rewritten as:

$$\prod=\{\sigma\}^T\{\varepsilon^e\}+\{\chi\}^T\{\varepsilon^p\} \tag{12}$$

where χ is the dissipative stress.

Equation (11) could be also written in term of total strain

$$\prod=\{\sigma\}^T\{\varepsilon\}+\{\chi-\sigma\}^T\{\varepsilon^p\} \tag{13}$$

From Ziegler's orthogonality condition [15].

$$\{\chi-\sigma\}^T=0 \tag{14}$$

Then equation (12) reduces to:

$$\prod=\{\sigma\}^T\{\varepsilon\} \tag{15}$$

The free energy in the soil domain surrounding the disc can then calculated by:

$$F=\int_{r_0}^{r_m} r\int_0^{2\pi}\frac{1}{2}\{\sigma\}^T\{\varepsilon^e\}d\theta dr \tag{16}$$

where r_m is the radial distance at which the displacement becomes insignificant.

The difference between external and internal energy must be zero

$$\delta\prod=\delta\prod{}_{int}-\delta\prod{}_{ext}=0 \tag{17}$$

The energy equation can be differentiated to obtain a governing equation of the form:

$$\delta\prod=\sum\left[A_k(U,V,W)\delta u_k{}^{(1)}\right]+\left[B_k(U,V,W)\delta u_k{}^{(2)}\right]+\left[C_k(U,V,W)\delta v_k{}^{(1)}\right]+\left[D_k(U,V,W)\delta v_k{}^{(2)}\right]=0 \tag{18}$$

The governing equations for deformation can be obtained by collecting the coefficients of δu_k and δv_k

$$\frac{d^2u_k}{dr^2}+\frac{1}{r}\frac{du_k}{dr}-\left(\frac{2+k^2(1-2v)-2v}{2(1-v)}\right)\frac{u_k}{r^2}+\frac{k(3-4v)v_k}{2(1-v)r^2}-\frac{k}{2(1-v)}\frac{1}{r}\frac{dv_k}{dr}=F_{rk}{}^p \tag{19}$$

$$\frac{d^2v_k}{dr^2}+\frac{1}{r}\frac{dv_k}{dr}-\left(\frac{1+2k^2(1-v)-2v}{(1-2v)}\right)\frac{v_k}{r^2}+\frac{k(3-4v)u_k}{(1-2v)r^2}+\frac{k}{(1-2v)}\frac{1}{r}\frac{du_k}{dr}=F_{\theta k}{}^p \tag{20}$$

where

$$F_{rk}{}^{p} = \frac{d\hat{\varepsilon}_{rk}{}^{p}}{dr} + \frac{v}{(1-v)}\left(\frac{d\hat{\varepsilon}_{\theta k}{}^{p}}{dr} + \frac{d\hat{\varepsilon}_{zk}{}^{p}}{dr}\right) + \frac{(1-2v)}{(1-v)}\left(\frac{\hat{\varepsilon}_{rk}{}^{p} - \hat{\varepsilon}_{\theta k}{}^{p} - k\hat{\varepsilon}_{r\theta k}{}^{p}}{r}\right) \tag{21}$$

$$F_{\theta k}{}^{p} = 2\frac{d\hat{\varepsilon}_{r\theta k}{}^{p}}{dr} + \frac{2kv}{(1-2v)}\left(\frac{\hat{\varepsilon}_{rk}{}^{p} + \hat{\varepsilon}_{zk}{}^{p}}{r}\right) + \frac{2k(1-v)}{(1-2v)}\left(\frac{\hat{\varepsilon}_{\theta k}{}^{p}}{r}\right) + 4\frac{\hat{\varepsilon}_{r\theta k}{}^{p}}{r} \tag{22}$$

It should be emphasised that equations (19)–(22) are one-dimensional equations (a function of r only) and this is only possible by using the Fourier series approximation for the displacements and the strains and their integration properties. Therefore, the 2D problem is transformed into a much simpler system of 1D equations.

Boundary conditions

For a rough interface between the rigid disc and the surrounding soil (if a lateral load is associated with θ = 0), the displacement at the boundary is given by:

$$u_k = \begin{pmatrix} \Delta & \text{if } k = 1 \\ 0 & \text{if } k \neq 1 \end{pmatrix} \text{at } r = r_0 \tag{23}$$

$$v_k = \begin{pmatrix} \Delta & \text{if } k = 1 \\ 0 & \text{if } k \neq 1 \end{pmatrix} \text{ at } r = r_0 \tag{24}$$

$$u_k = v_k = 0 \text{ at } r = r_m \tag{25}$$

where Δ is the lateral deformation of the disc.

Solution procedure

In elastoplastic problems, an iterative procedure is required to solve the governing equations (19 and 20). At first, the right hand of equations (19) and (20), which represent the plasticity terms, can be taken to be zero (i.e. we start by assuming elastic response). Then the components u_k and v_k and the displacements are calculated, and the total strains are calculated using equation (3). The stresses are then calculated from the total strain. If the stresses lie outside the assumed yield criteria, the stresses are corrected, and the plastic components of strains are calculated. Once the plastic strain is calculated, then the harmonic coefficients are obtained using the fitted method (see Appendix) and substituted in equations (21) and (22) to calculate $F_{rk}{}^{p}$ and $F^{p}_{\theta k}$. The plastic terms in equations (19) and (20) are then updated, and the equations are solved for new displacements. These calculations need to be iterated until the difference between the new and old displacements become within a certain tolerance.

The lateral force per unit length can then be obtained by integrating the stresses around the disc as follows:

$$P = -\int_0^{2\pi} r_0(\sigma_r \cos\theta - \sigma_{r\theta} \sin\theta)d\theta \tag{26}$$

Validation of the proposed solution

Elastic material

For a purely elastic material, $F_{rk}{}^{p} = 0$ and $F_{\theta k}{}^{p} = 0$ and problem is simplified so that equations (18) and (19) are needed to be solved only for k = 1.

Baugelin et al. [16] and Klar and Osman [4] obtained solutions for displacements for a laterally loaded disc in an elastic material. These solutions are obtained by using the general solution of Airy Stress function derived by Mitchell [17]. The elastic analytical solution required an assumption that the displacements vanished at a distance r_m away from a rigid disc representing a pile of radius r_o (Fig. 1), without which infinite displacements would result. Baguelin et al. [16] also suggested expressions relating r_m to the embedded length of the pile for different pile head fixity conditions. Assuming the pile is bonded to an intact elastic soil, expressions for radial and circumferential stresses in the soil around the pile are:

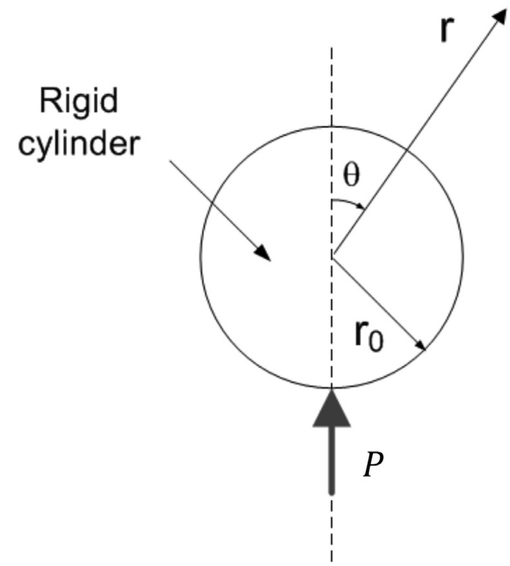


Fig. 1. Laterally loaded disc in elasto-plastic material.

$$\sigma_r = -\frac{P}{4\pi r_0}\frac{1}{(1-\nu)}\left[(3-2\nu)\frac{r_0}{r} - \frac{1}{1+(r_0/r_m)^2}\left(\frac{r_0}{r}\right)^3 + \frac{1}{(3-4\nu)}\frac{1}{1+(r_m/r_0)^2}\frac{r}{r_0}\right]\cos\theta \tag{27}$$

$$\sigma_\theta = \frac{P}{4\pi r_0}\frac{1}{(1-\nu)}\left[(1-2\nu)\frac{r_0}{r} - \frac{1}{1+(r_0/r_m)^2}\left(\frac{r_0}{r}\right)^3 - \frac{3}{(3-4\nu)}\frac{1}{1+(r_m/r_0)^2}\frac{r}{r_0}\right]\cos\theta \tag{28}$$

It should be noted here that the classical solid mechanics sign convention is adopted here where compressive stresses are negative. The corresponding radial displacement is given by:

$$U = \frac{P}{16\pi G}\frac{1}{1-\nu}\left[(3-4\nu)\ln\left(\frac{r_m}{r}\right)^2 - \left(\frac{r_0}{r}\right)^2\frac{r_m{}^2 - r^2}{r_m{}^2 + r_0^2} - \frac{(4\nu-1)}{(3-4\nu)}\frac{r_m{}^2 - r^2}{r_m{}^2 + r_0^2}\right]\cos\theta \tag{29}$$

where G is the shear modulus of the soil and ν is Poisson ratio.

Baguelin et al. [16] also developed solutions that take into account the disturbance of the soil in the vicinity of the pile. Expressions relating r_m to the embedded length of the pile were suggested for different pile head fixity conditions. In elatic solutions, it is helpful to form dimensionless groups of relevant parameters, rather than investigate how the solution is affected by the variation of each individual soil parameter. Following the technique of dimensional analysis, it can be shown that:

$$\frac{UG}{P} = f_1\left(\nu, \frac{R}{r_0}\right) \tag{30}$$

Similarly, the incremental effective stress can be expressed as:

$$\frac{\sigma r_0}{P} = f_2\left(\nu, \frac{R}{r_0}\right) \tag{31}$$

Fig. 2 shows that the solution for the displacements and the stresses obtained from the current analysis, which is based on variational method (Equations (19) and (20)), is identical to that obtained from Airy Stress function [16].

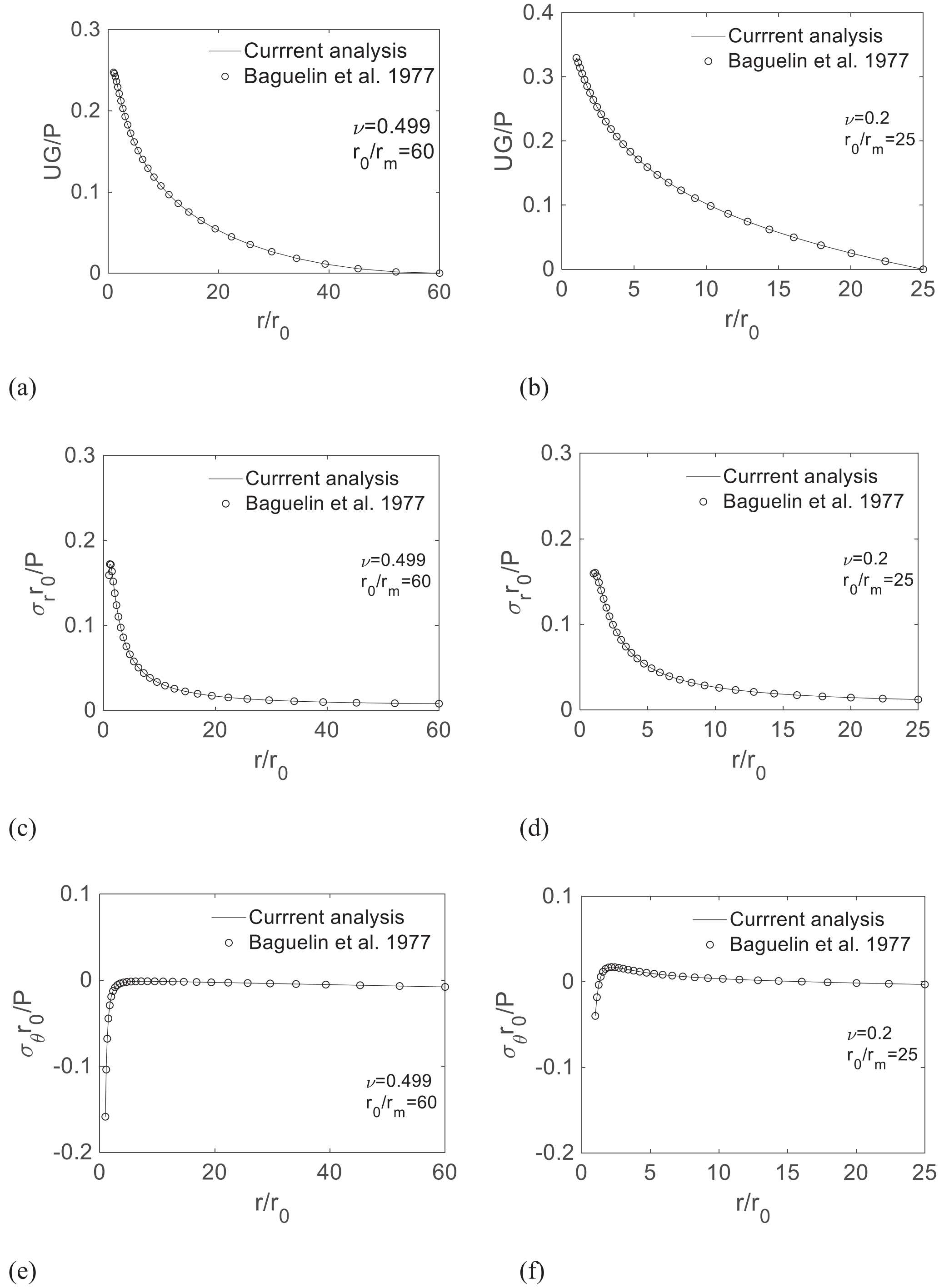


Fig. 2. Elastic analysis: comparison with previously published solutions.

Elasto-plastic material

There are no closed-form expressions for the displacement and stress for the case of a laterally loaded rigid disc in an elastoplastic material. Therefore, the comparison is made with finite element FE analysis. The FE analysis was carried out using ABAQUS software [18]. The soil was modelled using eight-node plane-strain elements. Because of the symmetry, only half of the problem was modelled. Full-fixity boundary conditions are assumed at the far-field while displacements normal to the line of symmetry are prevented. The FE mesh consisted of 504 elements and 1605 nodes, with details of the finite element mesh shown in Fig. 3.

The rigid disc is taken to have a radius of 1 m embedded in elastoplastic soil with soil's Young modulus $E_s = 10$ MPa and Poisson ratio $\upsilon = 0.499$.

For simplification, the comparison is made with Von-Mises yield criteria:

$$F = \sigma - 2k \tag{32}$$

where σ denotes the second stress invariant, and $2k$ is the uniaxial yield stress.

The plastic strain can be estimated as

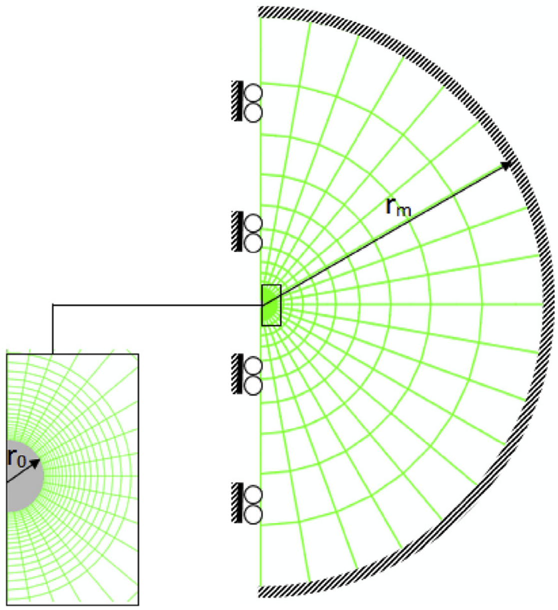


Fig. 3. Finite element mesh (r_m = 60 m, r_0 = 1 m).

$$\varepsilon^p = \lambda \frac{dF}{d\sigma} \tag{33}$$

By using a radial return algorithm, the Lagrangian multiplier λ can be evaluated as

$$\overline{\Delta\lambda} = \frac{\sigma - 2k}{3G} \tag{34}$$

F represents the plastic potential function, and $\overline{\Delta\lambda}$ is equal to the deviatoric plastic strain equivalent.

Fig. 4 shows the rigid disc load-displacement curve obtained from the analytical solution is in consistent with the finite element results. The analytical solutions are obtained using trucated Fourier series with L = 20 (equation (2)). Contours of deviatoric stresses (Mises stresses) are shown in Fig. 5. Fig. 5a shows the contours obtained using ABAQUS finite element software while Fig. 5b shows the results from the analytical solutions. The contours pattern are in general agreements between the FE

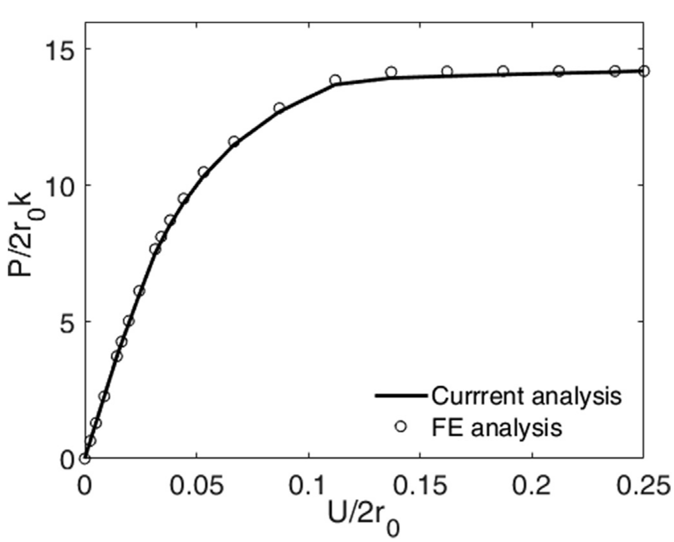
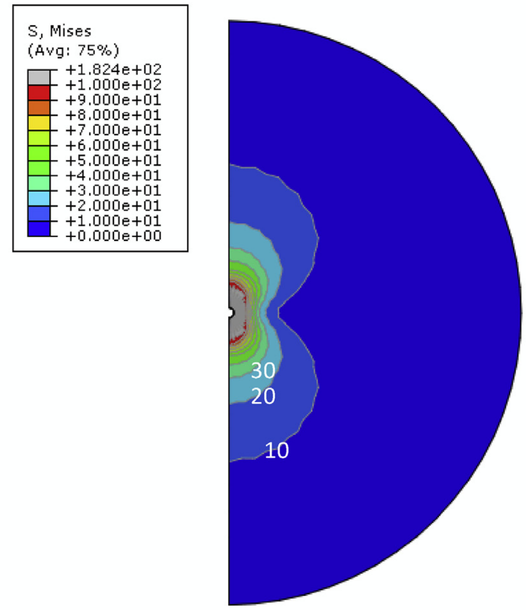
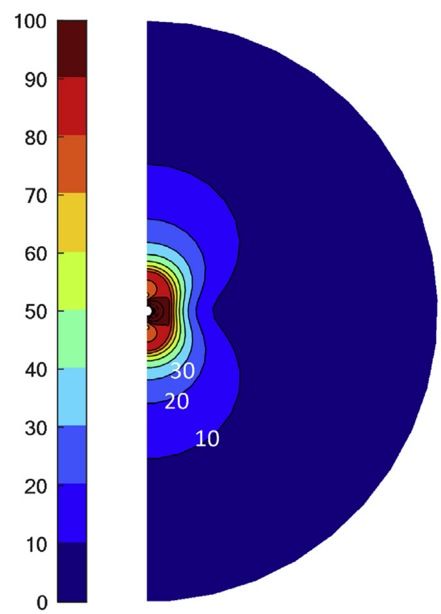


Fig. 4. The deflection of a laterally-loaded disc in elasto-plastic material (θ = 0°).



(a)



(b)

Fig. 5. Contours of deviatoric stress: (a) ABAQUS FE analysis (b) Analytical solution.

and the analytical solution. The minor discrepancies might be due to the algorithm adopted by ABAQUS for plotting the contours. Then values are then extrapolated to the nodes and averaged then the contours plotted using the nodal values. It should be borne in mind that standard FE yields a stress field that is discontinuous across interelement boundaries.

Conclusions

An analytical solution has been developed to predict the response of a rigid in an elastoplastic soil. The displacement components in cylindrical coordinates are represented by Fourier series. The governing differential equations are obtained using the principle of minimizing the potential energy. The results were validated against previously published solutions and against finite element analysis.

The key advantage of the presented methodology in this paper is that solutions for two and three-dimensional problems are obtained by solving one-dimensional governing equations. Therefore, the analytical solutions presented in this note could be regarded as an extension to Vlasov variation methods to elastoplastic problems.

The validity of the proposed analytical solution is demonstrated here by analysing a laterally loaded pile assuming plane strain assumptions. However, the methodology could be extended the 3D analysis of piles by rewriting the radial and circumferential displacements in term of Fourier series and follow the procedure illustrated in this note.

Credit author statement

Salma Hashem-Ali: Writing- Original draft preparation, Investigation, Ashraf Osman: Supervision, Writing- Reviewing and Editing, Conceptualization, Methodology

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Appendix

The fitted method is suitable to determine the harmonic coefficients (see Ref. [19] for example), by assuming x is known values, and X represents the unknown harmonic coefficients. Fourier series harmonic coefficients can be estimated by considering variable x which is a function of θ ($x = f(\theta)$). Then x can be given by

$$x = X^0 + X^1 \cos\theta + X^1 \sin\theta + X^2 \cos 2\theta + X^2 \sin 2\theta + \ldots + X^l \cos l\theta + X^l \sin l\theta + \ldots \tag{35}$$

Equation (35) can be rewritten in matrix form as

$$\begin{Bmatrix} x_1 \\ x_2 \\ x_3 \\ \\ x_{n-1} \\ x_n \end{Bmatrix} = \begin{bmatrix} 1 & \cos\theta_1 & \sin\theta_1 & \cdots & \cos L\theta_1 & \sin L\theta_1 \\ 1 & \cos\theta_2 & \sin\theta_2 & \cdots & \cos L\theta_2 & \sin L\theta_2 \\ 1 & \cos\theta_3 & \sin\theta_3 & \cdots & \cos L\theta_3 & \sin L\theta_3 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & \cos\theta_{n-1} & \sin\theta_{n-1} & \cdots & \cos L\theta_{n-1} & \sin L\theta_{n-1} \\ 1 & \cos\theta_n & \sin\theta_n & \cdots & \cos L\theta_n & \sin L\theta_n \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \\ \\ X^{\frac{n-1}{2}} \\ X^{\frac{n-1}{2}} \end{Bmatrix} \tag{36}$$

which can be rewritten as

$$x = [H]X \tag{37}$$

where L denotes the order of Fourier series, x is the vector of known values, X represents the vector of unknown harmonic coefficients and H is the harmonic transformation matrix.

Solutions have been derived for the special case of having known x, at equispaced θ values (see Fig. 6). The harmonic coefficients can be estimated from the following equations

$$X^0 = \frac{1}{n}\sum_{i=l}^{n} x_i \tag{38a}$$

$$X^k = \frac{2}{n}\sum_{i=l}^{n} x_i \cos k(i-l)\alpha \tag{38b}$$

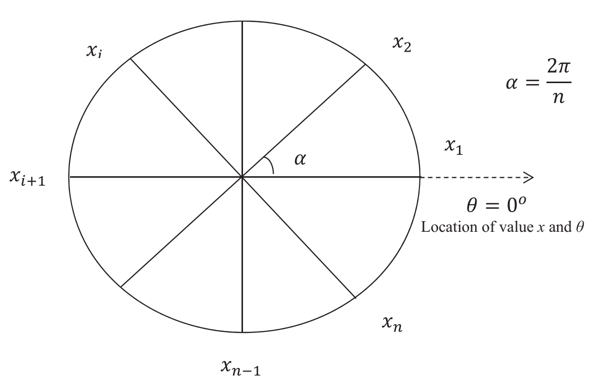


Fig. 6. Fitted method: locations of value x_i

$$X^k = \frac{2}{n}\sum_{i=l}^{n} x_i \sin k(i-l)\alpha \tag{38c}$$

References

[1] B.B. Broms, Lateral resistance of piles in cohesive soils, J. Soil Mech. Found. Div. ASCE 90 (2) (1964) 27–43.
[2] M.F. Randolph, G.T. Houlsby, The limiting pressure on a circular pile loaded laterally in cohesive soil, Geotechnique 34 (4) (1984) 613–623.
[3] C.M. Martin, M.F. Randolph, Upper bound analysis of lateral pile capacity in cohesive soil, Geotechnique 56 (2006) 141–145.
[4] A. Klar, A.S. Osman, Continuous velocity fields for the T-bar problem, Int. J. Numer. Anal. Methods Geomech. 2008 (32) (2008) 949–963.
[5] E. Winkler, Die Lehre Von Der Elastizitat Und Festigkeit, Dominicus, Prague, 1867.

[6] V.Z. Vlasov, U.N. Leontiev, Beams, Plates, and Shells on Elastic Foundation, Israel Program for Scientific Translations, Jerusalem, 1966 translated from Russian).
[7] R. Jones, J. Xenophontos, The Vlasov foundation model, Int. J. Mech. Sci. 19 (1977) 317–323.
[8] C.V.G. Vallabhan, Y.C. Das, Parametric study of beams on elastic foundations, J. Eng. Mech. Div. 114 (1988) 2072–2082.
[9] C.V.G. Vallabhan, Y.C. Das, Modified Vlasov model for beams on elastic foundations, J. Geotech. Eng. 117 (1991) 956–966.
[10] K. Sun, Laterally loaded piles in elastic media, J. Geotech. Eng ASCE 120 (8) (1994) 1324–1344.
[11] W.Y. Shen, C.I. Teh, Analysis of laterally loaded piles in soil with stiffness increasing with depth, J. Geotech. Geoenv. Eng., ASCE 130 (8) (2004) 878–882.
[12] K. Yang, R. Liang, Numerical solution for laterally loaded piles in a two-layer soil profile, J. Geotech. Geoenv. Eng., ASCE 132 (11) (2006) 1436–1443.
[13] D. Basu, R. Salgado, M. Prezzi, A continuum-based model for analysis of laterally loaded piles in layered soils, Geotechnique 59 (2) (2009) 127–140.
[14] T.J. Birchall, A.S. Osman, Response of a tunnel deeply embedded in a viscoelastic medium, Int. J. Numer. Anal. Methods Geomech. 36 (15) (2012) 1717–1740.
[15] H. Ziegler, An Introduction to Thermomechanics, second ed., North Holland, Amsterdam, 1983.
[16] F. Baguelin, R. Frank, Y.H. Said, Theoretical study of lateral reaction mechanism of pile, Geotechnique 27 (3) (1977) 405–434.
[17] J.H. Mitchell, On the direct determination of stress in an elastic solid with application to the theory of plates, Proc. Lond. Math. Soc. 1889 (31) (1889) 100–124.
[18] Dassault Systèmes, ABAQUS Users' Manual, Dassault Systèmes Simulia Corp., Providence, RI, USA, 2016.
[19] D.M. Potts, L. Zdravkovic, Finite Element Analysis in Geotechnical Engineering, Theory, Thomas Telford, 1999.